The derivation of the equation governing the vibrating string yields the one-dimensional wave equation. We derive this equation and then use the solution to this partial differential equation to model particular physical phenomena. As we observe the behavior of the one-dimensional wave given various boundary conditions, we integrate a discussion of how normal modes influence the shape of the string as it moves through time.

**Introduction**

Physics is often considered to be the branch of science that studies matter and energy. Physicists describe the relationships between matter and energy and how they affect one another through space and time; these relationships can be defined quantitatively through differential equations. Ordinary differential equations are frequently used to model physical phenomena, such as mechanical and electrical oscillations. However, in regards to practicality, partial differential equations are used more frequently as most systems involve more than one variable, generally time and displacement. Bernhard Riemann is quoted as stating

...partial differential equations are the basis of all physical theorems. In the theory of sound in gases, liquids, and solids, in the investigations of elasticity, in optics, everywhere partial differential equations formulate basic laws of nature which can be checked against experiments.

In addition to more complex systems, partial differential equations model vibrating strings, which we will study in detail through this paper. Specifically, we will first verify the well-known wave equation through its derivation and observe particular application of the equation.
One-Dimensional Wave Equation: Derivation

Let \( L \) be the length of a stretched string with ends fastened on the \( x \)-axis at \( x = 0 \) and \( x = L \). Also, let \( u(x, t) \) denote the transverse placement at \( t \geq 0 \) of the point on the string at position \( x \); specifically, \( u(x, 0) \) denotes the initial shape of the string. We want to determine the motion of the string by finding \( u(x, t) \) such that \( t > 0 \) and \( 0 < x < L \). Assume that vibration occurs when the string is displaced from the equilibrium then released, that the string is perfectly elastic so there is no resistance to movement, and transverse vibrations of the string are small and take place in the \( xu \)-plane, which contains the \( x \)-axis.

The constant mass density of the string is represented by \( \rho \), and \( \tau \) is the magnitude of the tension at equilibrium. We simplify the problem to assume that there are no external forces acting on the string except its tension, \( \tau \), with a constant tension throughout the motion. Consider two points on the string, say \( A > 0 \) and \( B > 0 \), located at \( x \) to \( x + \Delta x \) with tensions \( \tau_1 \) and \( \tau_2 \), respectively. Notice that \( \tau_1 \) and \( \tau_2 \) have the same magnitude \( \tau \) but differ in direction; here there is only motion in the vertical direction so we can disregard the notion of horizontal motion.

We allow \( \alpha \) to denote the angle formed by the tangent and horizontal axis at point \( A \), and \( \beta \) as the angle formed by the tangent and horizontal at point \( B \). Hence, the vertical components are \( -\tau \sin(\alpha) \) and \( \tau \sin(\beta) \), respectively. Now, applying Newton’s second law, \( F = ma \), we obtain

\[
-\tau \sin(\alpha) + \tau \sin(\beta) = ma
\]

\[
= (\rho)(\Delta x) \left( \frac{\partial^2 u}{\partial t^2} \right),
\]

noting that the second time derivative measures acceleration. For small angles \( \alpha \) and \( \beta \), \( \cos(\alpha) = 1 \) and \( \cos(\beta) = 1 \), and thus \( \sin(\alpha) \approx \tan(\alpha) \) and \( \sin(\beta) \approx \tan(\beta) \). Hence,

\[
-\tau \tan(\alpha) + \tau \tan(\beta) = (\rho)(\Delta x) \left( \frac{\partial^2 u}{\partial t^2} \right).
\]

Let \( t \) be fixed, then consider \( u(x, t) \) as a function of \( x \) alone. The slope of the tangent line is \( \frac{\partial u}{\partial x}(x, t) \), and hence \( \tan(\alpha) = \frac{\partial u}{\partial x}(x, t) \) and \( \tan(\beta) = \frac{\partial u}{\partial x}(x + \Delta x, t) \). Thus,

\[
\tau \left[ \frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right] = (\rho)(\Delta x) \left( \frac{\partial^2 u}{\partial t^2} \right)
\]

\[
\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \Delta x = \rho \left( \frac{\partial^2 u}{\partial t^2} \right).
\]

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As $\Delta x \to 0$, the left hand side approaches $\frac{\partial^2 u}{\partial t^2}(x, t)$. Let $c^2 = \frac{\tau}{\rho}$ denote the velocity $c$, such that $\tau$ has units of $\frac{\text{mass-length}}{\text{time}^2}$ and $\rho$ has units $\frac{\text{mass}}{\text{length}}$, so that $c^2$ has units $\frac{\text{length}^2}{\text{time}^2}$. Therefore, we obtain the one-dimensional wave equation for the free vibrations of the string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$  \hspace{1cm} (1)

**Solution of One-Dimensional Wave Equation**

**The Method of Separation of Variables**

We can obtain a solution to the wave equation given an arbitrary initial position, displacement, and velocity using the method of separation of variables. Assume a string is stretched along the x-axis with its endpoints fixed at $x = 0$ and $x = L$ where $L$ is the length of the string. Also, let $u(x, t)$ denote the position, at some time $t$, of the point $x$ on the string. We begin with the wave equation (1), and form an initial boundary value problem by considering the boundary and initial conditions

$$u(0, t) = 0 \text{ and } u(L, t) = 0, \quad t > 0 \hspace{1cm} (2)$$

$$u(x, 0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x, 0) = g(x), \quad 0 < x < L. \hspace{1cm} (3)$$

Notice that the boundary conditions (2) are held fixed at 0 throughout the duration of the movement of the string. Additionally, in the initial conditions (3), $f(x)$ denotes the initial displacement and $g(x)$ denotes the initial velocity. This solution can be obtained using the method of separation of variables:

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x (b_n \cos \lambda_n t + b_n^* \sin \lambda_n t) \hspace{1cm} (4)$$

where

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \, dx, \hspace{1cm} (5)$$

$$b_n^* = \frac{2}{cn\pi} \int_{0}^{L} g(x) \sin \frac{n\pi}{L} x \, dx, \quad n \in \mathbb{N}, \hspace{1cm} (6)$$

and

$$\lambda = c \frac{n\pi}{L}$$

are determined by applying Fourier series half-range expansions.
Normal Modes

Notice that the solution to the one-dimensional wave equation is in summation notation. This clearly defines the solution to be the sum of the sine functions with increasingly small periods according to the values of $n$. Each iteration becomes a series of individual terms called the normal modes, illustrated in Figure 1. For any $n$, the string has identical lobes and divides the string into $n$ equidistant parts. The solution of the vibrating string is just the infinite sum of the normal modes, the elements that effect the shape of the string. Hence as $n$ increases the frequency also increases.

![Figure 1: The first, second, and third modes.](image)

Here the string has fixed ends and moves strictly vertically. When $n = 1$, we observe what is called the fundamental mode, shown in the first image of the figure below. However, the length of the string does not change as $n$ increases, and thus as $n$ increases the amplitude decreases as depicted in Figure 1.

Examples

Example 1: Vibration of a stretched string with fixed ends

Now that the general solution (4) to the wave equation (1) has been determined for fixed boundary conditions (2) and arbitrary initial conditions
(3), we can apply it to any stretched string with initial conditions to model its behavior.

Let’s first consider a string of length \( L = 1 \) fixed at \( x = 0 \) and \( x = 1 \). The string is set to vibrate from rest by releasing it from an initial triangular shape modeled by the function

\[
f(x) = \begin{cases} 
\frac{3}{10} x & \text{if } 0 \leq x \leq \frac{1}{3} \\
\frac{3}{20} (1 - x) & \text{if } \frac{1}{3} \leq x \leq 1 
\end{cases}
\]

We want to determine the subsequent motion of the string, given that \( c = \frac{1}{\pi} \). Since the general solution (4) to the boundary value problem (1, 2, 3) is dependent on \( b^* \) and \( b_n \), we must determine these two components in order to use the solution. Here, \( b^*_n = 0 \) because the string is initially at rest, and hence \( g(x) = 0 \). Then, \( b_n \) is defined in (5), where in this example \( L = 1 \). Hence we can substitute our initial function into the equation and integrate by parts to obtain

\[
b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx
= \frac{9}{10\pi^2} \sin \frac{n\pi}{3} n^2.
\]

Also, we determine that \( \lambda_n = c \frac{n\pi}{L} = \frac{1}{\pi} \left( \frac{n\pi}{1} \right) = n \), and thus

\[
u(x, t) = \frac{9}{10\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2} \sin n\pi x \cos nt
\]

is the solution to our boundary value problem.

Notice that when \( n = 3k \) the solution is a product of \( \sin(k\pi) \). Because of the cyclic nature of sine we find that we will see no contribution to the shape of the string from the normal modes when \( n = 3k \). This solution can be illustrated in MATLAB, given a range of values for \( x, t \) and \( n \). Figure 2 illustrates an amalgamation of still shots of this string while in motion, when \( n = 1 : 3 \). The fundamental mode and the second mode are clearly shaping the string as the \( x \)-axis is divided into two identical lobes and two equal parts, along the still shot of the solution when \( n = 2 \) as shown in Figure 1.

**Example 2: Vibration of a stretched string with fixed ends using two initial points of release.**

We note the behavior of the string using the conditions in Example 1, and now consider using similar conditions to observe how the string’s behavior will change when we initially release the string from two different positions. From the previous example, \( b^*_n = 0 \), \( L = 1 \), the string is fixed at \( x = \)
0 and \( x = 1, c = \frac{1}{\pi} \), and \( \lambda_n = n \) remains unchanged. However, one initial point of release is located at (.25, .1) and the other is located at (.75, .1). Hence, the string can be defined by the function

\[
    f(x) = \begin{cases} 
        \frac{2}{5}x & \text{if } 0 \leq x \leq \frac{1}{4} \\
        \frac{1}{10} & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\
        \frac{2}{5}(1 - x) & \text{if } \frac{3}{4} \leq x \leq 1.
    \end{cases}
\]

Here,

\[
    b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx 
= \frac{4}{5n^2\pi^2} \left( \sin \frac{n\pi}{4} + \sin \frac{3n\pi}{4} \right).
\]

Therefore, the complete solution is

\[
    u(x, t) = \frac{4}{5\pi^2} \sum_{n=1}^{\infty} \sin \frac{n\pi}{4} + \sin \frac{3n\pi}{4} \sin n\pi x \cos nt.
\]

Instinctively, we observe that when \( n = 4k \) the solution is a product of \( \sin(k\pi) \). As in Example 1, we find that these normal modes equal 0. However, if we recall the cyclic nature of sine we can observe that any other even values of \( n \) will also equal 0 as the numerator will have the form \( "1 + (-1)" \) or \( "(-1) + 1." \) So when \( n = 2k \), the solution is 0. Hence, the shape of the string will only be effected by the odd normal modes. As executed in the previous
example, this particular solution is substituted into the general solution then implemented using MATLAB to illustrate the changing dynamics of the string over time.

![Initial shape and still shots of motion](image)

Figure 3: Initial shape, and still shots of motion for Example 2.

In Figure 3 we illustrate the movement of the string through several still shots when $n = 1 : 3$. As we observe the behavior around the $x$-axis we observe the contribution from the first and third modes to the shape of the string. Recall, the third mode is shown in Figure 1, and has a small magnitude when compared to the fundamental mode.

**Example 3: Vibration of a stretched string with fixed ends and a non-zero initial velocity**

In one last effort to illustrate the various contributions from the normal modes, we return to Example 1 and now let the initial velocity be non-zero. Namely, let $g(x) = \frac{\pi}{40} x$. Because our initial velocity is no longer zero we are required to find the value of $b_n^*$. Recall, $b_n^*$ is defined by (6) and here $b_n^* = \frac{2}{n} \int_0^1 g(x) \sin(n\pi x) dx$. Following similar steps from Example 1 we find that $b_n^* = \frac{(-1)^{n+1}}{20n^2}$. Thus, our solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{9}{10n^2} \sin \frac{n\pi}{3} \sin n\pi x \cos nt + \frac{(-1)^{n+1}}{20n^2} \sin nt.$$ (8)

We note that (8) is identical to (7) but with the additional term from $b_n^*$. Recall from our discussion of Example 1 that there is no contribution to the shape of the string when $n$ is a multiple of three in the first term of
(8). However, the second term of the sum implies that these values of \( n \) will now yield contribution. This defines the impact of the initial velocity on the movement of the string. Following the algorithms of the previous examples, the particular solution is substituted into the general solution and then implemented using MATLAB to illustrate the change in the string as it moves through time. It is straightforward to observe the change in motion between strings in Figure 4. The initial velocity effects the entire motion of the string while maintaining the spatial influence of the normal modes.

![Figure 4: Still shots of the motion for Example 1 with no initial velocity, and then with non-zero initial velocity (Example 3).](image)

In a similar manner as in the previous examples, we observe the fundamental and second modes in both illustrations in Figure 4, with minor contribution from the third mode in the second image. Although the general movement of the string changes drastically with a non-zero initial velocity, the movement of the string is just a summation of the normal modes, weighted differently based on the initial positions and velocities.

**Conclusion**

Throughout this paper, we have taken a step-by-step approach in deriving the one-dimensional wave equation. We then discussed the significance of normal modes and their influences on the shape of the string as it pertains to the Fourier representation of the solutions. We illustrated these concepts with three examples to allow comparison between initial conditions. When we analytically solved the wave equation in each of these examples, we found the solutions to be infinite sums of the normal modes, the general shapes which dictate the movement of the string. Although this was a basic set of examples, in general, boundary value problems such as these are used
to model systems that involve more than one variable, specifically time and
displacement, often in multiple directions, with a vast number of applications.

References

[1] Asmar, Nakhlé H., *Partial differential equations and boundary value prob-